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NON STANDARD ANALYSIS AND THE COMPACTIFICATION OF GROUPS

BY
YORAM HIRSCHFELD

ABSTRACT
The lattice of compactifications of a given group is related to the set lattice of a family of subgroups of the enlargement. We investigate the relations between these lattices and obtain a description of the universal compactification in terms of the neighbourhoods of the identity with finite index.

In [5] A. Robinson used non standard analysis to obtain some compactifications of groups and rings, as homomorphic images of the enlargement. Developing his ideas we obtain a more systematical connection between compactifications and a family of sub-groups of $^*G$. These groups are characterized and a direct way to obtain the compactification as a quotient group using the corresponding group is described. With this correspondence the lattice structure of the family of compactifications corresponds to the set lattice structure of the family of subgroups (see 1.4 for a precise summary).

This method gives rise to a new standard description of the universal compactification which starts with the sets of finite index. This is done in Section 2. In particular — the universal compactification of a ring with a unit is the inverse limit of its finite quotient rings.

We assume familiarity with the non standard characterizations of topological groups and filters as described in [3]. Also some familiarity with methods of non standard analysis is assumed since we shall not repeat in details the usual arguments.

Our terminology follows [3] but it is adapted to the common use in non standard analysis. Thus $\mu(F)$ is the monad of F and not its nucleus (although it is still a nuclear set). In notations we accept the common use. Thus $x \in {}^*A$ and not $x \in \hat{A}$ or $x^* \in A$. As usual ${}^0x$ is the standard point near x (assuming that such a point exists).

We begin with a fixed group G and a topology T on the group. T will denote the neighbourhood system of the identity which gives a complete description of the topology. If μ is the monad of T we denote the topological group by $\langle G, T\rangle$ or by $\langle G, \mu\rangle$. When additional groups will be considered the notations will be $\langle G, \mu(G)\rangle$, $\langle H, \mu(H)\rangle$, etc. If in such a case $\langle H, \mu(H)\rangle$ is said to be the compactification of $\langle G, \mu(G)\rangle$ then a map $f: G \to H$ is implicitly assumed.

We enlarge a universe that includes G and all its compactifications (it is difficult to figure out which compactifications might be omitted and why). We assume that the enlargenment is saturated enough to have the following properties:

0.1 If F is a filter on G and f is a mapping then $\mu(f(F)) = f(\mu(F))$ [3; 5.1.10];

0.2 If F and F' are filters on G then $\mu(F)\cdot\mu(F')$ is the monad of the filter $H = \{AB \mid A \in F \text{ and } B \in F'\}$.

It is always the case that $\mu(F)\cdot\mu(F') \subset \mu(H)$. If now $x \in \mu(H)$ then the following is finitely satisfied on the pairs $\langle A, B\rangle$ in $F \times F'$: $X \subseteq A \wedge Y \subseteq B \wedge x \in XY$. Therefore by saturation there is also an infinitesimal pair $X \in {}^*F$ $Y \in {}^*F'$ and $x \in XY$ so that $x \in \mu(F)\cdot\mu(F')$.

Finally it is worth mentioning that the whole discussion may be done in terms of filters on G rather then of compact monads. It involves however some difficulties and does not seem more natural to us.

1. The lattice of compactifications

We begin by specifying a subset λ of ${}^*G$ which plays a main role in our discussion.

1.1 First we define an equivalence relation on ${}^*G: x \sim y$ if for every standard set A, $x \in {}^*A$ iff $y \in {}^*A$. This is indeed an equivalence relation and the equivalence classes are the monads of ultrafilters [3; 5.1.5].

Let λ now be the set

$$\lambda = \{xy^{-1} \mid x \sim y\}$$

(λ will be fixed throughout the paper). It is easy to see that $x \sim y$ iff $x^{-1} \sim y^{-1}$ and that $x \sim y$ iff $ax \sim ay$ for every standard element a. Therefore we have the following:

1.2. Lemma.

i) $\lambda = \lambda^{-1}$

ii) $a\lambda a^{-1} = \lambda$ *for every standard element* a.

1.3. Definition. A compact monad for the topology μ is a subgroup ν of $^*G$ which satisfies the following conditions:

i) $\lambda \cup \mu \subset \nu$;

ii) ν is *nuclear* (i.e.: ν is the monad of a filter);

iii) ν is a normal subgroup of $^*G$.

1.4. Given a compact monad ν we denote by i_ν the natural map from $^*G$ onto $^*G/\nu$ and by T_ν the image of the filter generated by ν, i.e.

$$T_\nu = \{i_\nu(^*A) \mid \nu \subset {}^*A\}.$$

We intend to show:

a) $\langle ^*G_\nu, T_\nu\rangle$ is a compactification of $\langle G, T\rangle$;

b) every compactification is obtained in this manner. Moreover, every isomorphism class of compactifications is represented by a unique compact monad.

c) One compact monad is included in the other iff there is an epimorphism from the second compactification onto the first one (preserving the image of G). Thus the supremum of any number of compactifications corresponds to the intersection of the monads. The infimum of two compactifications corresponds to the product of the monads.

1.5. Theorem. *Let f be a (continuous) homomorphism of $\langle G, \mu(G)\rangle$ onto a dense subgroup of the compact group $\langle H, \mu(H)\rangle$. Let ν be $f^{-1}(\mu(H))$. Then ν is a compact monad and $\langle ^*G/\nu, T_\nu\rangle$ is isomorphic to $\langle H, \mu(H)\rangle$ over the embedding of G, by the map $j(x\nu) = {}^0(f(x))$.*

Proof. $f: {}^*G \to {}^*H$ is an algebraic homomorphism. Since $\mu(H)$ is a normal subgroup of $^*H$ [4; 8.1.8], ν is a normal subgroup of $^*G$. ν is nuclear as the inverse image of a nuclear set [3; 5.1.9] and $\mu(G) \subset \nu$ because of the continuity of f. Next assume that $x \sim y$. There is a standard element $a \in H$ such that $f(x) \in \mu(a)$. If $f(y) \not\in \mu(a)$ then there is a neighbourhood W of a in H such that $f(y) \not\in {}^*W$ and $f(x) \in {}^*W$. Therefore $x \in {}^*f^{-1}(W)$ and $y \not\in {}^*f^{-1}(W)$. Assuming $x \sim y$ this is impossible so that also $f(y) \in \mu(a)$. But then $f(xy^{-1}) = f(x)(f(y))^{-1} \in \mu(a)(\mu(a))^{-1} \subset \mu(H)$.

Next we define the map $j: {}^*G \to H$ by $j(x) = {}^0(f(x))$. j is well defined: as H is a compact Hausdorff space every point is near a unique standard point. $j(xy) = {}^0[f(xy)] = {}^0[f(x)f(y)] = {}^0[f(x)]\,{}^0[f(y)]$ so that j is an algebraic homomorphism. Since $f(G)$ is dense in H, j is onto H. Clearly ν is the kernel of j so that $^*G/\nu$ is algebraically isomorphic to H via the map $j(x\nu) = {}^0[f(x)]$. The isomorphism is over the embedding of G since for a standard element a we have $j(i_\nu(a)) = j(a) = {}^0(f(a)) = f(a)$.

It remains to show that under this isomorphism T_ν is carried to the neighbourhood filter T of the identity of H and vice versa.

Let V be a neighborhood in T. We show a set in T_ν whose image is included in $\bar{V}$. This suffices for one direction since the closed neighbourhoods in T form a base for T. We have $\mu(H) \subset {}^*V$ so that $\nu \subset f^{-1}({}^*V) = {}^*[f^{-1}(V)]$. Thus ${}^*[f^{-1}(V)]\,\nu \in T_\nu$ and

$$j({}^*[f^{-1}(V)]\nu) = \{{}^0[f(x)] \mid x \in f^{-1}({}^*V)\} \subset \{{}^0y \mid y \in {}^*V\} \subset \bar{V}.$$

On the other hand if $\nu \subset {}^*A$ then $f^{-1}(\mu(H)) \subset {}^*A$. Therefore there exists a neighbourhood $V \in T$ such that $f^{-1}(V) \subset A$. We show that $j^{-1}(V) \subset {}^*A\nu$: If $x\nu \in j^{-1}(V)$ then $j(x\nu) \in V$ and ${}^0[f(x)] \in V$. Therefore $f(x) \in {}^*V \cdot \mu(H)$ and $x \in f^{-1}({}^*V) \cdot \nu \subset {}^*A\nu$.

We denote by $\nu(H)$ the compact monad which was obtained from the compactification $\langle H, \mu(H)\rangle$ in Theorem 1.5.

1.6. THEOREM. *For every compact monad ν there is a compactification $\langle H, \mu(H)\rangle$ such that $\nu = \nu(H)$.*

PROOF.

a) We assume first that e is the only standard element in ν. Then ν is the monad of a Hausdorff topology on G.

The monad of a point x is $x\nu$ and since $\nu \supset \lambda$ it includes the monad of the ultrafilter generated by x. Therefore the topology is precompact [3; 7.4] and G can be densely imbedded in a compact group $\langle H, \mu(H)\rangle$ which induces on G the topology determined by ν. Thus $\nu = \mu(H) \cap {}^*G$ and $\nu = \nu(H) = f^{-1}(\mu(H))$ where f is the identity map.

b) In the general case we let K be the set of standard elements in ν. $K = \nu \cap G$ and ν is normal in ${}^*G$ so that K is normal in G. Let G_1 be the quotient group G/K and let g be the natural mapping of G onto G_1. We want to apply part (a) of the proof to G_1. G_1 is a topological group with the natural quotient topology, and $\mu(G_1) = g(\mu(G))$. We denote by ν_1 the set $g(\nu)$; then clearly $\nu_1 \supset \mu(G_1)$ and ν_1 is a normal subgroup in ${}^*G_1 = g({}^*G)$. ν_1 is nuclear by 0.1. To prove that ν_1 is a compact monad it remains therefore to show that $\nu_1 \supset \lambda(G_1)$. Given $x \in {}^*G$ let λx be the monad of the ultrafilter generated by x in G and let $\lambda g(x)$ be the monad of the ultrafilter generated by $g(x)$ in G_1. Assume now that $y \in \lambda x$. Then a separation of $g(x)$ from $g(y)$ by a standard set B would induce a separation between x and y by $g^{-1}(B)$. Therefore $g(\lambda x) \subset$

$\lambda g(x)$. But $g(\lambda x)$ is nuclear by 0.1 and since the monad of the ultrafilter $\lambda g(x)$ is a minimal nuclear set we conclude that $g(\lambda x) = \lambda g(x)$. If now $z \sim t$ in $^*G_1$ and $z = g(x)$ then there is also some $y \in \lambda x$ such that $t = g(y)$. Therefore $zt^{-1} = g(xy^{-1}) \in g(\nu) = \nu_1$. Therefore $\nu_1 \supset \lambda(G_1)$, and ν_1 is a compact monad.

To finish our proof we observe the following claims:

i) $^*K \subset \nu$;

ii) $g^{-1}(\nu_1) = \nu$.

If $\nu \subset {}^*V$ then $K \subset {}^*V$. But this implies that $K \subset V$ and $^*K \subset {}^*V$. Thus $^*K \subset \bigcap \{^*V \mid \nu \subset {}^*V\} = \nu$. This proves claim (i).

Next assume that $x \in g^{-1}(\nu_1)$ so that $g(x) \in \nu_1$. Since $g(\nu) = \nu_1$ there is some $\alpha \in \nu$ such that $g(x) = g(\alpha)$ and $g(x\alpha^{-1}) = e$. Hence $x\alpha^{-1} \in {}^*K \subset \nu$ and $x \in \nu \cdot \alpha = \nu$. This proves claim (ii).

Thus every standard element in ν_1 is the image of at least one standard element which must be in ν by claim (ii), and therefore in K. We conclude that e is the only standard element in ν_1 which is itself a compact monad and part (a) applies. Thus there is a compactification $\langle H, \mu(H)\rangle$ of $\langle G_1, \mu(G)\rangle$ for which $f^{-1}(\mu(H)) = \nu_1$. It is easy to see that $f \circ g$ turns H into a compactification for $\langle G, \mu(G)\rangle$ and by claim (ii) $(f \circ g)^{-1}(\mu(H)) = \nu$.

1.7. Corollary. *For every compact monad ν, $\langle {}^*G/\nu, T\nu\rangle$ is a compactification and all the compactifications are obtained in this manner.*

Proof. Given a compact monad ν we know by Theorem 1.6 that $\nu = \nu(H)$ for some compactification $\langle H, \mu(H)\rangle$ and that $\langle {}^*G/\nu, T\nu\rangle$ is isomorphic to $\langle H, \mu(H)\rangle$ by Theorem 1.5. Theorem 1.5 also assures that all the compactifications are thus obtained.

1.8. Theorem. *Let $\langle H, \mu(H)\rangle$ and $\langle H', \mu(H')\rangle$ be compactifications of G. If there exists an epimorphism of H onto H' which preserves the image of G then $\nu(H) \subset \nu(H')$.*

Proof. Let g and g' be the mappings of G into H and H' respectively, and let h be the epimorphism of H onto H'. Then $h^{-1}(\mu(H')) \supset \mu(H)$ since h is continuous. Therefore $g'^{-1}h^{-1}(\mu(H')) \supset g^{-1}(\mu(H))$. In other words $(h \circ g)^{-1}(\mu(H')) \supset \nu(H)$. But $h \circ g = g'$ and therefore $\nu(H') = g'^{-1}(\mu(H')) \supset \nu(H)$.

We have now a complete description of the relations between compact monads and compactifications. By 1.8 each isomorphism class of compactifications gives rise to a single compact monad ν, for which $^*G/\nu$ is in this class. Since for every compactification H in this class $\nu = \nu(H)$, we have also $\nu = \nu({}^*G/\nu)$. Therefore the correspondences $\nu \to {}^*G/\nu$ and $H \to \nu(H)$ are the inverses of each other and they identify compact monads with isomorphism classes of compactifications.

We note also the converse of 1.8:

1.9. THEOREM. *If $\nu \subset \nu'$ then there is an epimorphism of ${}^*G/\nu$ onto ${}^*G/\nu'$.*

PROOF. There is a natural map form ${}^*G/\nu$ onto ${}^*G/\nu'$ and it is easy to see that it is a topological epimorphism.

1.10. LEMMA.

i) *The intersection of a family of compact monads is a compact monad.*

ii) *The product of two compact monads is a compact monad.*

PROOF.

i) The intersection is a normal subgroup, includes λ and $\mu(G)$, and is nuclear by [3; 5.1.6].

ii) The product is a normal subgroup, includes λ and $\mu(G)$ and it is nuclear by 0.2.

1.11. Therefore, it is clear that the intersection of a family of compact monads represents the smallest compactifications which can still be mapped onto each of the compactifications in family. Similarly, the product of two compact monads represents the largest compactification which is the image of the two compactifications.

2. The Universal compactification

The compact monad for the universal compactification is simply the smallest such monad — the intersection of all compact monads. In this section we describe a (standard) procedure to obtain the filter which is associated with it.

2.1. We note first that if we are given a compact monad ν which is the monad of the filter F then F characterizes the compactification which can be constructed directly as follows (we analyze the standard version of Theorem 1.6): F describes a (not necessarily Hausdorf) topology on G. If K is the set of elements which belong to every member of F then the topology induced on G/K is precompact and its completion is the compactification whose compact monad is $\mu(F)$. Thus a standard construction of the filter whose monad is the universal compact monad is indeed a description of the universal compactification.

2.2. LEMMA. *If ν is a nuclear subgroup of $^*G$ such that $\lambda \subset \nu$ and such that $a\nu a^{-1} \subset \nu$ for every standard element a then ν is normal in G.*

PROOF. We denote by F the filter whose monad is ν. It sufices to show that $x\nu x^{-1} \subset {}^*V$ for every $x \in {}^*G$ and $V \in F$. So we assume that x and V are given. Let B be an infinitesimal member of F. Then $B^3 \subset {}^*V$ since ν is a group and therefore $W^3 \subset V$ also for some $W \in F$. Replacing W by $W \cap W^{-1}$ which is again in F we may assume that $W = W^{-1}$. Next let A be an infinitesimal set in the ultrafilter generated by x. Then $AA^{-1} \subset {}^*W$ since $\lambda \subset \nu$. Therefore also $C \cdot C^{-1} \subset W$ for some standard set C that contains x. We chose a standard element $c \in C$ and we have $x \in {}^*C \subset {}^*Wc$ and $x^{-1} \in c^{-1*}W$. Finally $cBc^{-1} \subset c\nu c^{-1} = \nu \subset {}^*W$ so that $cUc^{-1} \subset W$ also for some standard U in F. Adding everything up we have $x\nu x^{-1} \subset ({}^*Wc)U(c^{-1*}W) \subset {}^*W^3 \subset {}^*V$.

2.3. *Thus we may replace condition (iii) of* 1.3 *by:*
iii)′ $a\nu a^{-1} \subset \nu$ for every standard element a in G.
We describe first the filter which belongs to the universal compactification using the set λ of 1.1.

2.4. Let F be the filter generated by $\lambda \cup \mu(G)$. Let F' be the collection of sets V with the following property: there are sequences $\{V_i\}$ such that $V_0 = V$, $V_i \supset V_{i+1}^2$ and $V_i \in F$ for $i = 1, 2, \cdots$. F' has the following properties:

a) F' is a filter: if $\{V_i\}$ is a sequence for V and $\{W_i\}$ is a sequence for W then $\{V_i \cap W_i\}$ is a sequence for $V \cap W$;

b) if $V \in F'$ then $V^{-1} \in F$: if $\{V_i\}$ is a sequence for V then $\{V_i^{-1}\}$ is a sequence for V^{-1}. $V_i^{-1} \in F$ since $\lambda = \lambda^{-1}$;

c) similarly $aVa^{-1} \in F'$ whenever $V \in F'$. This follows from the fact that $a\lambda a^{-1} \subset \lambda$.

2.5. THEOREM. *$\mu(F')$ is the compact monad of the universal compactification.*

PROOF. Clearly $\mu(F') \supset \lambda \cup \mu(G)$. if $x, y \in \mu(F')$ and $V \in F'$ then there is some $V_1 \in F'$ such that $V_1^2 \subset V$. By 2.4 (b), $y^{-1} \in \mu(F')$ and $xy^{-1} \in {}^*V_1^2 \subset {}^*V$. Since V was arbitrary in F' this shows that $\mu(F')$ is a group. By 2.4 (c) and 2.3 this is a normal subgroup of $^*G$. We conclude that $\mu(F')$ is a compact monad.

Next let ν be any compact monad which generates the filter U. For the universality of $\mu(F')$ it suffices to show that every set in U is in F'. Since ν is a group it is easy to construct for every $V_0 \in U$ a sequence $\{V_i\}$ such that $V_{i+1}^2 \subset V_i$ and $V_i \in U$ for $i = 1, 2, \cdots$. Since $\nu \supset \mu(G) \cup \lambda$ this proves that $V_0 \in F'$.

2.6. Finally we want to give a completely standard description of the universal compactification filter F' of 2.4. A subset V of G is of *finite index* if there are $b_1, \cdots, b_n$ such that $Vb_1 \cup \cdots \cup Vb_n = G$. Let L be the collection of sets V with the following property: there is a sequence of symmetric neighbourhoods of the identity $\{V_i\}$ such that $V_1 \subset V$, $V_{i+1}^2 \subset V_i$ and V_i has finite index for $i = 1, 2, \cdots$.

THEOREM. $L = F'$.

PROOF. We show first that every set in F' has a finite index. Assume that $V \in F'$ and for every $a_1, \cdots, a_n$ there is some $b \in G$ such that $b \notin Va$. Then by the main property of enlargements there is some $x \in {}^*G$ such that $x \notin {}^*Va$ for every standard $a \in G$. On the other hand since $\lambda \subset {}^*V$ there is a set A such that $x \in {}^*A$ and $AA^{-1} \subset V$ (because every infinitesimal set in the ultrafilter generated by x satisfies this). Therefore for every $a \in A$ we have $x \in {}^*Va$, a contradiction. Let now $\{V_i\}$ be a sequence of sets like in 2.4. Then $\{V_i \cap V_i^{-1}\}$ is a sequence of symmetric sets with finite index (since they are again in F') and clearly they are neighbourhoods of the identity as they include $\mu(G)$. Thus $V_0 \in L$ and $F' \subset L$.

On the other hand assume that $\{V_i\}$ is a sequence for V like in 2.6. We show that $V_i \in F$ for all i so that $V \in F'$ and $L \subset F'$. Since V_{i+1} has a finite index there are elements $a_1 \cdots a_n$ such that $V_{i+1}a_1 \cup \cdots \cup V_{i+1}a_n = G$. Given any $x \in {}^*G$ one of the sets $V_{i+1}a_j$ is in the filter generated by x. Therefore $(V_{i+1}a_j)(V_{i+1}a_j)^{-1} = V_{i+1}^2 \subset V_i$. Therefore the monad λx of the filter generated by x satisfies $\lambda x(\lambda x)^{-1} \subset {}^*V_i$.

In particular every subgroup of finite index is in L. If L has a base which consists of such normal subgroups then the universal compactification can be represented as the inverse limit of the finite quotient groups. This follows from [1; III, 7.3, cor. 2]. We want to show that this is always the case for rings with a unit element.

2.7. Let R be a ring with a unit. The same theory leads us to the notion of a compact monad which must be a two sided ideal rather then a normal subgroup.

THEOREM. *The universal compactification of R is the inverse limit of the finite rings R/I where I ranges over all the ideals with finite index in R.*

PROOF. Let L be the universal compactification filter and let ν be its monad. By [1; III, 7.3, cor. 2] it suffices to show that L has a base which consists of ideals. Let $\langle H, \mu(H)\rangle$ be the universal compactification. Since $\nu = f^{-1}(\mu(H))$ every set

V in F includes a set $f^{-1}(W)$ where W is a neighbourhood of 0 in H. Therefore it suffices to show that each neighbourhood of 0 in H includes an open ideal. Also $f(1)$ is a unit in H so that if $H \neq 0$ then by [2, theor. 8] H is totally disconnected. Thus 0 has a base by clopen sets and we can assume that W is clopen.

Let U be an infinitesimal neighbourhood of 0 in H. It suffices to show that the *ideal generated by U is included in $^*W$. This ideal is the collection of internal sums $\Sigma_{i=1}^{\pi} a_i v_i b_i$ where π is in $^*N$, $a_i, b_i \in {}^*R$ and $v_i \in U$. Assume that there is such a sum which is not in $^*W$ and let $\alpha = a_1 v_1 b_1 + \cdots + a_{\pi+1} v_{\pi+1} b_{\pi+1}$ be the shortest such sum. Put $\beta = a_1 v_1 \beta_1 + \cdots + a_\pi v_\pi b_\pi$. Then $\beta \approx a$ for some standard $a \in H$ and $a \in W$ since $\beta \in W$ and W is closed. But $a_{\pi+1} v_{\pi+1} b_{\pi+1}$ is in $\mu(H)$ which is an ideal so that $\alpha \in \mu(a) \subset {}^*W$ since W is also open; which is a contradiction. This proves the theorem.

2.8. Finally we like to mention that Lemma 2.2 gives a clear connection between the compactifications of a given topology $\mu(G)$ and those of the discrete topology. If ν is a compact monad for the discrete topology then $\nu \cdot \mu(G)$ is a compact monad for $\mu(G)$ since (i), (iii) and (iii)′ are satisfied. Clearly all the compactifications of $\mu(G)$ are obtained in this way and the universal compactification is obtained by taking ν to be the universal compactification for the discrete topology.

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TEL AVIV UNIVERSITY
TEL AVIV, ISRAEL